

## AN EMBEDDING THEOREM FOR ABELIAN MONOIDAL CATEGORIES

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ABSTRACT. We show that, with some technical conditions, an abelian category can be embedded into the category of bimodules over a ring. The case of semisimple rigid monoidal categories is studied in more detail.

## INTRODUCTION

The problem of finding an embedding theorem for abelian monoidal categories is motivated on one hand by the Freyd-Mitchell full embedding theorem and on the other hand by Deligne's and Doplicher-Roberts' theories of "abstract Tannakian-Krein's duality" [3, 5].

By definition, an embedding from an abelian category to another is a faithful functor which sends non-zero objects to non-zero objects. According to Freyd and Mitchell, a small abelian category admits an exact full embedding into the category of modules over a ring [10]. This important theorem allows one for example to treat finite diagrams in an abelian category as diagrams of modules.

In [3], Deligne shows that, under certain technical conditions, an abelian symmetric rigid monoidal category admits an exact monoidal embedding into the category of modules over a commutative ring. Hence, by Tannaka-Krein's duality, such a category is equivalent to the category of representation of a groupoid [3, Theorem 1.12]. Doplicher and Roberts study the case of compact groups and obtain an analogous result for  $C^*$ -categories [5].

With the birth of quantum groups [6], the theory of monoidal category has a new motivation. The "symmetric" condition turns out to be too strong and is replaced by a weaker one, the "braided" condition. The problem of generalizing Deligne's and Doplicher-Roberts' results to braided categories is interesting. Yet, one does not know what can be a target for such an embedding.

The situation seems to be simpler if we drop the symmetry, i.e., to find an embedding for abelian monoidal categories. A natural candidate for the target category is the category of bimodules over a ring. In this paper we show that any small monoidal category with exact tensor product admits a right exact monoidal embedding into the category of bimodules over a ring. In particular, a small abelian rigid monoidal category admits an exact monoidal embedding (Theorem 3.2).

Unfortunately, this embedding theorem does not seem to help solve the problem for braided categories, for, according to Schauenburg [12], the category of bimodules over a ring is never braided unless the basic ring is a field.

Let me briefly explain the main idea of the paper. To find an embedding for a small abelian monoidal category, we first extend it to a bigger monoidal category which is cocomplete and has an injective cogenerator, namely, a Grothendieck monoidal category. Then we extend the latter category to a module category, say over a ring  $R$ . Finally, we construct a monoidal functor from a module category with a monoidal structure to a bimodule category with the usual tensor

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product. The construction of the last functor will be given in Section 2. In Section 3 we explain how to extend the tensor product on a small abelian monoidal category to a monoidal structure on a module category containing this small abelian monoidal category. In the last section we consider an application to the special case of semisimple categories. An explicit embedding is described. As a consequence, we show that a semisimple symmetric monoidal category with simple unit object is Tannakian (Corollary 4.3).

Throughout the paper, the tensor product over a ring  $R$  is denoted by  $\otimes$ .  $\otimes$  also denotes the tensor product in an abstract monoidal category when no confusion may appear, otherwise, we use the signs  $\odot$  or  $\square$ . The category of right  $R$ -modules (resp. left  $R$ -modules or  $R$ - $R$ -bimodules) is denoted by  $\mathbf{Mod}_R$  (resp.  ${}_R\mathbf{Mod}$  or  ${}_R\mathbf{Mod}_R$ ).  $\mathbf{Hom}_R$  (resp.  ${}_R\mathbf{Hom}$  or  ${}_R\mathbf{Hom}_R$ ) denotes the set of homomorphisms of right  $R$ -modules (resp. left  $R$ -modules or  $R$ - $R$ -bimodules).

## 1. ABELIAN MONOIDAL CATEGORIES

**1.1. Monoidal categories.** Let  $\mathbf{A}$  be a category. A monoidal structure on  $\mathbf{A}$  consists of the following data: a bifunctor  $\otimes : \mathbf{A} \times \mathbf{A} \rightarrow \mathbf{A}$ ,  $(X, Y) \mapsto X \otimes Y$ , called tensor product, an object  $I$ , called unit object, for which

1. there exists a natural isomorphism  $\alpha$  between the functors  $(- \otimes -) \otimes -$  and  $- \otimes (- \otimes -)$ ,  $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$ , called associator, such that the following diagram commutes:

$$\begin{array}{ccc} ((X \otimes Y) \otimes Z) \otimes U & \xrightarrow{\alpha_{X,Y,Z} \otimes 1} & (X \otimes (Y \otimes Z)) \otimes U \xrightarrow{\alpha_{X,Y \otimes Z,U}} X \otimes ((Y \otimes Z) \otimes U) \\ \alpha_{X \otimes Y,Z,U} \downarrow & & \downarrow 1 \otimes \alpha_{Y,Z,U} \\ (X \otimes Y) \otimes (Z \otimes U) & \xrightarrow{\alpha_{X,Y,Z \otimes U}} & X \otimes (Y \otimes (Z \otimes U)) \end{array} \quad (1)$$

2. there exist natural isomorphisms of functors  $- \otimes I$ ,  $I \otimes -$  and the identity functor:  $\rho_X : X \otimes I \rightarrow X$  and  $\lambda_X : I \otimes X \rightarrow X$ , called right and left units, such that the following diagram commutes:

$$\begin{array}{ccc} (X \otimes I) \otimes Y & \xrightarrow{\alpha_{X,I,Y}} & X \otimes (I \otimes Y) \\ \rho_X \otimes 1 \searrow & & \swarrow 1 \otimes \lambda_Y \\ & X \otimes Y & \end{array} \quad (2)$$

$(\mathbf{A}, \otimes, I, \alpha, \lambda, \rho)$  is called a monoidal category. In the case, when  $\alpha, \lambda, \rho$  are identity morphisms, we have a strict monoidal category. In the general case, the associator  $\alpha$  allows one to speak of a tensor product of many objects  $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ , without specifying the order in which the tensor product is applied (MacLane's coherence theorem [9, VII,2]).

Using (1) and 2, we can show that the following diagrams commute:

$$\begin{array}{ccc} (I \otimes X) \otimes Y & \xrightarrow{\alpha_{I,X,Y}} & X \otimes (I \otimes Y) \\ \lambda_X \otimes 1 \searrow & & \swarrow \lambda_{X \otimes Y} \\ & X \otimes Y & \end{array} \quad \begin{array}{ccc} (X \otimes Y) \otimes I & \xrightarrow{\alpha_{X,Y,I}} & X \otimes (Y \otimes I) \\ \rho_X \otimes Y \searrow & & \swarrow 1 \otimes \rho_Y \\ & X \otimes Y & \end{array} \quad (3)$$

and that  $\lambda_I = \rho_I$  (cf. [11]).

From the definition of a bifunctor, we have

$$f \otimes g = (1 \otimes g) \circ (f \otimes 1) = (f \otimes 1) \circ (1 \otimes g). \quad (4)$$

In particular, using the isomorphism  $\lambda_I = \rho_I$ , we have, for  $r, s \in \text{End}(I)$ :

$$\begin{array}{ccccc}
 I \otimes I & \xrightarrow{r \otimes 1} & I \otimes I & \xrightarrow{s \otimes 1} & I \otimes I \\
 \downarrow \rho_I & & \downarrow \rho_I & & \downarrow \lambda_I \\
 I & \xrightarrow{r} & I & \xrightarrow{s} & I.
 \end{array} \tag{5}$$

Therefore, using the equation (4), we have  $r \circ s = s \circ r$ . Thus,  $\text{End}(I)$  is an abelian group with respect to composition. Further, this group acts on any set  $\text{Hom}(X, Y)$  from the left and the right by means of the isomorphism  $\lambda$  and  $\rho$ :

$$r \cdot f := \lambda_Y(r \otimes f) \lambda_X^{-1}, \quad f \cdot r := \rho_Y(f \otimes r) \rho_X^{-1}. \tag{6}$$

We have  $1_I \cdot f = f \cdot 1_I = f$ .

**1.2. The internal homs.** Each object  $X$  in  $\mathbf{A}$  defines a functor  $X \otimes - : \mathbf{A} \longrightarrow \mathbf{A}$ ,  $Y \longmapsto X \otimes Y$ . If this functor has a right adjoint, the right adjoint will be denoted by  $\text{rhom}(X, -)$ . We have, by definition, a natural isomorphism

$$\text{Hom}(X \otimes Y, Z) \cong \text{Hom}(Y, \text{rhom}(X, Z)), \quad \forall Y, Z. \tag{7}$$

The functor  $\text{lhom}$  is define analogously by

$$\text{Hom}(Y \otimes X, Z) \cong \text{Hom}(Y, \text{lhom}(X, Z)), \quad \forall Y, Z. \tag{8}$$

The category  $\mathbf{A}$  is called left closed (resp. right closed or closed) if the functor  $\text{lhom}(X, -)$  (resp.  $\text{rhom}(X, -)$  or both functors) is defined for any  $X \in \mathbf{A}$ . The general theory of adjoint functors (cf. [10, Corollary V.3.2]) gives us the following criteria of closedness

**Lemma 1.1.** *Assume that  $\mathbf{A}$  is a cocomplete category with a generator. Then the tensor product on  $\mathbf{A}$  is closed if and only if it commutes with colimits.*

In the general case, the functor  $\text{lhom}(X, -)$  and  $\text{rhom}(X, -)$  preserves colimits, whenever the latter is defined. If the category is closed, then we can also speak of the (contravariant) functors  $\text{lhom}(-, Z)$  and  $\text{rhom}(-, Z)$ .

**Lemma 1.2.** *Assume that  $\mathbf{A}$  is closed. Then the functors  $\text{lhom}(-, Z)$  and  $\text{rhom}(-, Z)$  preserve colimits (i.e. sending colimits to limits) whenever the latter is defined.*

*Proof.* We have, for any  $Y \in \mathbf{A}$ ,

$$\begin{aligned}
 \text{Hom}(Y, \text{rhom}(\varinjlim X_i, Z)) &\cong \text{Hom}(\varinjlim X_i \otimes Y, Z) \\
 &\cong \text{Hom}(\varinjlim (X_i \otimes Y), Z) \\
 &\cong \varprojlim \text{Hom}(X_i \otimes Y, Z) \\
 &\cong \varprojlim \text{Hom}(Y, \text{rhom}(X_i, Z)) \\
 &\cong \text{Hom}(Y, \varprojlim \text{rhom}(X_i, Z)),
 \end{aligned}$$

here we use the fact that  $\text{Hom}(Y, -)$  preserves limits. Since the above isomorphisms hold for any  $Y$ , we conclude that

$$\text{rhom}(\varinjlim X_i, Z) \cong \varprojlim \text{rhom}(X_i, Z).$$

The case of  $\text{lhom}(-, Z)$  is treated quite analogously. ■

**1.3. Rigid objects.** Setting  $Z = I$  and  $Y = \text{lhom}(X, I)$  in (8), we obtain a morphism  $\text{ev}_X : \text{lhom}(X, I) \otimes X \rightarrow I$ , which corresponds to the identity morphism in  $\text{Hom}(\text{lhom}(X, I), \text{lhom}(X, I))$ . It is obvious that  $(\text{lhom}(X, I), \text{ev}_X)$  is universal with this property. By definition, a left dual to an object  $X$  is a pair  $(X^*, \text{ev}_X : X^* \otimes X \rightarrow I)$  such that there exists a morphism  $\text{db}_X : I \rightarrow X \otimes X^*$  making the following diagrams commutative:

$$\begin{array}{ccc} X & \xrightarrow{\text{db}_X \otimes 1} & (X \otimes X^*) \otimes X \\ \downarrow \text{id} & & \downarrow \alpha_{X, X^*, X} \\ X & \xleftarrow{\text{id}_X \otimes \text{ev}_X} & X \otimes (X^* \otimes X) \end{array} \quad \begin{array}{ccc} X^* & \xrightarrow{\text{id} \otimes \text{db}_X} & X^* \otimes (X \otimes X^*) \\ \downarrow \text{id}_{X^*} & & \downarrow \alpha_{X^*, X, X^*}^{-1} \\ X & \xleftarrow{\text{ev}_X \otimes \text{id}_{X^*}} & (X^* \otimes X) \otimes X^* \end{array} \quad (9)$$

The left dual, if it exists, is uniquely determined up to an isomorphism. Moreover, we have a natural isomorphism

$$\text{lhom}(X, Z) \cong Z \otimes X^*. \quad (10)$$

In particular, the functor  $\text{lhom}(X, -)$  exists if  $X$  has a left dual. In this case, the diagrams in (9) also imply that the functor  $X \otimes -$  has a left adjoint:  $X^* \otimes -$ , consequently  $X \otimes -$  commutes with limits.

The definition of a right dual  ${}^*X$  to  $X$  is similar. Analogous assertions hold for objects having right dual. An object in  $\mathbf{A}$  is called rigid if it possesses left and right duals. The category  $\mathbf{A}$  is called rigid if its objects are rigid.

**1.4. Monoidal functors.** Let  $(\mathbf{A}, \otimes, \alpha, \rho, \lambda)$  and  $(\mathbf{A}', \otimes', \alpha', \rho', \lambda')$  be monoidal categories. A functor  $F : \mathbf{A} \rightarrow \mathbf{A}'$  is called monoidal functor if there exists a natural isomorphism  $\xi_{X, Y} : F(X) \otimes' (Y) \rightarrow F(X \otimes Y)$  and an isomorphism  $\eta : I' \rightarrow F(I)$ , such that the following diagrams are commutative:

$$\begin{array}{ccc} (F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{\xi \otimes' 1} & F(X \otimes Y) \otimes' F(Z) \xrightarrow{\xi} F((X \otimes Y) \otimes Z) \\ \downarrow \alpha' & & \downarrow F(\alpha) \\ F(X) \otimes' (F(Y) \otimes' F(Z)) & \xrightarrow{1 \otimes' \xi} & F(X) \otimes' F(Y \otimes Z) \xrightarrow{\xi} F(X \otimes (Y \otimes Z)) \end{array} \quad (11)$$

$$\begin{array}{ccc} I' \otimes' F(X) & \xrightarrow{\eta \otimes 1} & F(I) \otimes' F(X) \\ \downarrow \lambda' & & \downarrow \xi \\ F(X) & \xleftarrow{\lambda'} & F(I \otimes X) \end{array} \quad \begin{array}{ccc} F(X) \otimes' I' & \xrightarrow{1 \otimes \eta} & F(X) \otimes' F(I) \\ \downarrow \rho' & & \downarrow \xi \\ F(X) & \xleftarrow{F(\rho)} & F(X \otimes I) \end{array}$$

If an object  $X$  from  $\mathbf{A}$  is rigid then its image  $F(X)$  is also rigid.

**1.5. Abelian monoidal categories.** A monoidal category  $(\mathbf{A}, \otimes)$  is called abelian monoidal if it is abelian and the tensor product is an additive bifunctor.

In this case,  $K := \text{End}(I)$  is a commutative ring and  $\text{Hom}(X, Y)$  becomes  $K - K$ -bimodule, for any objects  $X, Y$ . Notice that the two actions of  $K$  do not generally coincide and  $\mathbf{A}$  is therefore not necessarily  $K$ -linear.

From the discussion in 1.2, if the functor  $\text{lhom}(X, -)$  is defined then it is left exact and the functor  $X \otimes -$  is right exact. The same holds for  $\text{rhom}(X, -)$  and  $- \otimes X$ . If  $\mathbf{A}$  is closed then the contravariant functor  $\text{lhom}(-, Z)$  and  $\text{rhom}(-, Z)$  are also left exact. If an object  $X$  has left dual then the functor  $X \otimes -$  is left exact; hence exact. In particular, if  $X$  is rigid then all the mentioned above functors are exact. Since  $X^* = \text{lhom}(X, I)$ , for a short exact sequence of

rigid objects  $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$  its dual sequence  $0 \longrightarrow Z^* \longrightarrow Y^* \longrightarrow X^* \longrightarrow 0$  is also exact.

## 2. TENSOR STRUCTURES ON MODULE CATEGORIES

Let  $R$  be a ring and  $\mathbf{Mod}_R$  be the category of right modules over  $R$ . Let  $\odot$  be a tensor product on  $\mathbf{Mod}_R$  with associator  $\alpha$  and left-, right units  $\lambda, \rho$ . We assume that the tensor product is (left and right) closed. That is, (cf 1.1) the functors  $-\odot X$  and  $X\odot -$  possess right adjoints. Consequently, these functors preserves colimits.

Consider a functor  $-\odot X : \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_R$ , which preserves colimits. By a theorem of Watts [14], there exists a left action of  $R$  on  $R\odot X$  making it an  $R-R$ -bimodule, such that the functor  $-\odot X$  is naturally equivalent to the functor  $-\otimes (R\odot X)$ :  $Y \longmapsto Y \otimes (R\odot X)$ . That is, there is a natural isomorphism

$$\theta_X(Y) : Y \otimes (R\odot X) \longrightarrow Y\odot X. \quad (12)$$

Throughout this section,  $\otimes$  denotes the tensor product over  $R$ . For the case  $X = R$ , we shall call the action of  $R$  on  $T := R\odot R$  the first left action of  $R$  on  $T$ , to distinguish with the second action defined subsequently.

Explicitly, the left action of  $R$  on  $R\odot X$  is given as follows

$$-\odot X : R \cong \mathbf{End}_R(R) \longrightarrow \mathbf{End}_R((R\odot X)). \quad (13)$$

Since  $\odot$  is biadditive, we see that  $\theta_X(Y)$  is also natural on  $X$ . Hence  $R\odot - : X \longmapsto R\odot X$  is a functor from the category  $\mathbf{Mod}_R$  to  ${}_R\mathbf{Mod}_R$ , commuting with colimits. Applying Watts' theorem again, we have an equivalence of the functors  $-\odot R$  and  $-\otimes T$  with a left action of  $R$  on  $T$ , called the second left action:

$$\mu_X : R\odot X \longrightarrow X \otimes_2 T, \quad (14)$$

where the subindex 2 indicates that the second left action of  $R$  on  $T$  is used to define the tensor product.

By its definition, the second left action commutes with the other actions of  $R$  on  $T$ , making  $T$  an object in  $(R\otimes_Z R)\mathbf{Mod}_R$ . And have an  $R$ -linear natural isomorphism

$$c_{X,Y} := \theta_Y(X) \circ (\mathrm{id}_X \otimes \mu_Y) : X \otimes_2 (Y \otimes_1 T) \longrightarrow X\odot Y. \quad (15)$$

The associator  $\alpha$  induces an  $R$ -linear natural isomorphism

$$\begin{aligned} \alpha'_{X,Y,Z} &:= (\mathrm{id} \otimes c_{Y,Z})c_{X,Y\odot Z}^{-1}\alpha_{X,Y,Z}c_{X\odot Y,Z}(c_{X,Y} \otimes \mathrm{id}) : \\ &(X \otimes_2 (Y \otimes_1 T)) \otimes_2 (Z \otimes_1 T) \longrightarrow X \otimes_2 ((Y \otimes_2 (Z \otimes_1 T)) \otimes_1 T) \end{aligned} \quad (16)$$

where the action of  $R$  is induced from the right action on  $T$  (indicated by a dot). Analogously, we have  $R$ -linear natural isomorphisms  $\lambda'$  and  $\rho'$

$$\lambda'_X := \lambda_X \circ \xi_{I,X} : I \otimes_2 (X \otimes_1 T) \longrightarrow X, \quad (17)$$

$$\rho'_X := \rho_X \circ \xi_{X,I} : X \otimes_2 (I \otimes_1 T) \longrightarrow X. \quad (18)$$

Thus, we have defined data for a monoidal structure on  $\mathbf{Mod}_R$ , namely, the tensor product of two module  $X, Y$  is  $X \otimes_2 (Y \otimes_1 T)$ , the associator is  $\alpha'$  and the left and right units is  $\lambda'$  and  $\rho'$ . Using the naturality of  $c$ , we can show that these data define a monoidal structure on  $\mathbf{Mod}_R$ . More explicitly, using routine diagram chasing, we can show the following lemma.

**Lemma 2.1.** *Let  $(A, \otimes, I, \alpha, \rho, \lambda)$  be a monoidal category. Let  $\odot$  be another bifunctor  $A \times A \longrightarrow A$ , which is equivalent to  $\otimes$  by means of a natural isomorphism  $\xi$ . Then  $\odot$  together with the isomorphism  $\alpha', \rho'$  and  $\lambda'$  defined as in (16), (17) and (18), define another monoidal structure on  $A$ . ■*

Setting  $X = Y = Z = R$  in (16), we obtain an isomorphism of  $R \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} R - R$ -bimodules

$$\alpha_{R,R,R} : \underset{\underset{1}{\circ}}{\underset{\underset{2}{\bullet}}{\bullet}} T \otimes_2 \underset{\underset{3}{\circ}}{\underset{\underset{1}{\bullet}}{\bullet}} T \longrightarrow \underset{\underset{3}{\circ}}{\underset{\underset{1}{\bullet}}{\bullet}} T \otimes_1 \underset{\underset{2}{\bullet}}{\bullet} T. \quad (19)$$

where  $\bullet, \circ, *$  denote the different actions of  $R$  on the source and the target of  $\Psi$  they will be referred to as the first, second and third left action of  $R$ . The right action is indicated by  $\cdot$ . The following lemma shows that  $\alpha'$  can be restored from this isomorphism.

**Lemma 2.2.** *Let  $P, Q$  be  $R - S$ -bimodules. Then a natural transformation  $c : - \otimes P \longrightarrow - \otimes Q$  of functors  $\text{Mod}_R \longrightarrow \text{Mod}_S$  is given by a  $R - S$ -bimodule homomorphism  $c = c_R : P \longrightarrow Q$ .*

*Proof.* We have the following commutative diagram:

$$\begin{array}{ccc} M \otimes P & \xrightarrow{c_M} & M \otimes Q \\ \downarrow f & & \downarrow f \otimes \text{id}_Q \\ N \otimes P & \xrightarrow{c_N} & N \otimes Q \end{array}$$

For  $N = M = R$  and a morphism  $f_s : R \longrightarrow R$  or right  $R$ -modules,  $f_s(r) := sr$ , we have

$$c(sp) = c_R(s \otimes p) = s(c_R(1 \otimes p)) = sc(p).$$

That is  $c$  is a left  $R$ -module morphism. By definition,  $c$  is a right  $S$ -module morphism, hence it is an  $R - S$ -bimodule morphism. Let now  $M = R$  and  $N$  arbitrary. For  $n \in N$ , choose  $f_n : R \longrightarrow N$ ,  $f_n(s) = ns$ , thus  $f$  is a morphism of right  $R$ -modules. Then we have, plugging  $f$  and  $c_N$  in the above diagram,  $c_N(n \otimes p) = (f \otimes \text{id}_Q)c_R(1 \otimes p) = n \otimes c(p)$ . ■

Setting in (18)  $X = R$  and  $Y = I$ , we have a natural isomorphism

$$\rho' := \rho \circ c(R, I) : I \otimes_1 T \longrightarrow R. \quad (20)$$

Analogously, we have a natural isomorphism

$$\lambda' := \lambda \circ c(I, R) : I \otimes_2 T \longrightarrow R. \quad (21)$$

The coherent constraints (1), (2), imply the following condition for the isomorphisms  $\alpha', \lambda', \rho'$ :

$$(\alpha' \otimes_1 \text{id}_T)(\text{id}_T \otimes_1 \alpha')(\alpha' \otimes_2 \text{id}_T) = (\text{id}_T \otimes_3 \alpha')(\text{id}_T \otimes_2 \alpha'), \quad (22)$$

$$(\lambda' \otimes_1 \text{id}_T)(\rho' \otimes_2 \text{id}_T) = \text{id}_I \otimes_1 \alpha'. \quad (23)$$

For any right  $R$ -modules  $M, N$ , we have a sequence

$$\begin{aligned} \text{Hom}_R(M, N) &\xrightarrow{\text{id}_M \otimes -} {}_2\text{Hom}_{R-R}(M \otimes_1 T, N \otimes_1 T) \xrightarrow{\text{id}_I \otimes -} \text{Hom}_R(M, N) \\ f &\longmapsto f \otimes \text{id}_T \longmapsto \text{id}_I \otimes f \otimes \text{id}_T = f \end{aligned} \quad (24)$$

here  ${}_2\text{Hom}_R(M \otimes_1 T, N \otimes_1 T)$  denotes the set of  $R - R$ -bimodule morphisms, the left actions of  $R$  on whose source and target are given by the second left action of  $R$  on  $T$ . According to the isomorphism in (17) the composition of the above morphisms is an isomorphism, hence the map

$$\text{Hom}_R(M, N) \longrightarrow {}_2\text{Hom}_R(M \otimes_1 T, N \otimes_1 T) \quad (25)$$

is injective.

Consider now the functor  $\omega : \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_R$ ,  $X \longmapsto X \otimes_1 T$ , where the left action of  $R$  on  $X \otimes_1 T$  is given by the second left action of  $R$  on  $T$ . We have a natural isomorphism

$$\begin{aligned} \xi_{X,Y} := \alpha'_{R,X,Y} : \quad & \omega(X) \otimes \omega(Y) = (X \otimes_1 T) \otimes_2 (Y \otimes_1 T) \\ & \xrightarrow{\alpha_{R,X,Y}} (X \otimes_2 T) \otimes_1 T = \omega(X \otimes_2 (Y \otimes_1 T)). \end{aligned} \quad (26)$$

**Lemma 2.3.** *With the notation as above,  $(\omega, \xi)$  is a monoidal functor from  $(\mathbf{Mod}_R, \odot)$  to  $({}_R\mathbf{Mod}_R, \otimes)$ .*

*Proof.* One has to check hexagon identity for  $\xi$ . Since in  ${}_R\mathbf{Mod}_R$ , the tensor product is strict, the hexagon can be reduced to a pentagon. It turns out that the commutativity of this pentagon is precisely the coherence condition of  $\alpha'$ , which holds by Lemma (2.1). ■

*Remark.* Although our monoidal category  $(\mathbf{Mod}_R, \odot)$  is not assumed to be strict. The functor  $\omega$  maps it into a strict monoidal category.

**Theorem 2.4.** *Let  $\odot$  be a monoidal structure on  $\mathbf{Mod}_R$ . Then the functor  $\omega : \mathbf{Mod}_R \longrightarrow {}_R\mathbf{Mod}_R$ ,  $X \longmapsto X \otimes_1 T$  is a monoidal right exact embedding.*

*Proof.* We have seen that  $\omega$  is monoidal, faithful, right exact. It remains to show that  $\omega(X) \not\cong 0$  whenever  $X \not\cong 0$ . That is,  $X \odot R \not\cong 0$  for all  $X \not\cong 0$ . Let  $X$  be such that  $X \odot R \cong 0$ . Then

$$0 = \mathrm{Hom}(X \odot R, X) \cong \mathrm{Hom}(R, \mathrm{rhom}(X, X)).$$

Therefore  $\mathrm{rhom}(X, X) = 0$ . But then  $\mathrm{Hom}(X, X) \cong \mathrm{Hom}(I, \mathrm{rhom}(X, X)) = 0$ . Thus,  $X \cong 0$ . ■

Let now  $\mathbf{A}$  be a monoidal category which, as an abelian category, is cocomplete with a progenerator (i.e. small projective generator [10]), and, as a monoidal category, is closed. Let  $P$  be a progenerator of  $\mathbf{A}$ . Then  $\mathbf{A}$  is equivalent to  $\mathbf{Mod}_R$ , where  $R = \mathrm{End}(P)$ , by the functor  $F = \mathrm{Hom}(P, -)$ . Since  $F$  is an equivalence, it carries the monoidal structure on  $\mathbf{A}$  over to  $\mathbf{Mod}_R$ . Thus, we have

**Corollary 2.5.** *Let  $(\mathbf{A}, \odot)$  be an abelian monoidal category, cocomplete and closed and with a progenerator. Then the functor  $\omega : X \longmapsto \mathrm{Hom}(P, X \odot P)$  is a monoidal functor from  $\mathbf{A}$  to  ${}_R\mathbf{Mod}_R$ , where  $R = \mathrm{End}(P)$ . This functor is a right exact embedding. It is exact if  $P$  is left flat with respect to the tensor product on  $\mathbf{A}$ . An analogous assertion holds for the functor  $X \longmapsto \mathrm{Hom}(P, P \odot X)$ .*

*Remark.* Since  $X \otimes_1 T \cong X \odot R$ . The functor  $\omega$  is exact if and only if  $R$  is flat with respect to the tensor product  $\odot$ . This is not always the case. Take for example the category  ${}_S\mathbf{Mod}_S$ , where  $S$  is a ring not flat over  $\mathbb{Z}$ . Then for  $R = S^{\mathrm{op}} \otimes_{\mathbb{Z}} S$ ,  ${}_S\mathbf{Mod}_S$  is equivalent to  $\mathbf{Mod}_R$ . The tensor product is taken over  $S$ , therefore  $M \otimes_S R = M \otimes_S (S^{\mathrm{op}} \otimes_{\mathbb{Z}} S) \cong M \otimes_{\mathbb{Z}} S$ . Thus  $R$  is not flat.

### 3. AN EMBEDDING THEOREM FOR SMALL ABELIAN MONOIDAL CATEGORIES

Using the result of the previous section, we show in this section that a small abelian monoidal category with exact tensor product can be embedded in the category of bimodules over a ring. Our tactic is to embed  $\mathbf{C}$  in a bigger category which is cocomplete with a projective generator.

**3.1. The category  $\text{Ind-C}$ .** A category  $\mathbf{I}$  is called a filtering category if to every pair  $i, i'$  of objects from  $\mathbf{I}$  there exists an object  $i''$ , such that  $\text{Hom}(i, i'')$  and  $\text{Hom}(i', i'')$  are both not empty, and for every pair of morphisms  $f, f' : i \longrightarrow i'$ , there exists a morphism  $g : i' \longrightarrow i$  equalizing them, i.e.  $gf = gf'$ .

Let  $\mathbf{C}$  be an abelian category. The category  $\text{Ind-C}$  of ind-objects of  $\mathbf{C}$  consists of functors  $\mathbf{X} : \mathbf{I} \longrightarrow \mathbf{C}$ , where  $\mathbf{I}$  is any small filtering category. Alternatively, denoting  $X_i := \mathbf{X}(i), i \in \mathbf{I}$ , an ind-object of  $\mathbf{C}$  is a directed system indexed by a small filtering category  $\mathbf{I}$ . For two objects  $\mathbf{X} = \{X_i\}_{i \in \mathbf{I}}$  and  $\mathbf{Y} = \{Y_j\}_{j \in \mathbf{J}}$ , their hom-set is

$$\text{Hom}(\mathbf{X}, \mathbf{Y}) := \varprojlim_i (\varinjlim_j \text{Hom}(X_i, Y_j)). \quad (27)$$

The following lemma will be usefull when dealing with hom-sets of ind-objects.

**Lemma 3.1.** (cf [2, Appendix, Cor. 3.2]) *A morphism  $f : \mathbf{X} \longrightarrow \mathbf{Y}$  can be represented, up to isomorphism, by a small filtering system of morphisms  $\{f_i : X_i \longrightarrow Y_i\}_{i \in \mathbf{I}}$ .*

$\mathbf{C}$  is fully embedded in  $\text{Ind-C}$  by a constant functor. On the other hand,  $\text{Ind-C}$  is fully embedded in  $\text{Fun}(\mathbf{C}^{\text{op}}, \text{Set})$ . For an ind-object  $\mathbf{X} = \{X_i\}_{i \in \mathbf{I}}$ , define the functor

$$\mathbf{L}_{\mathbf{X}} : \mathbf{Y} \longrightarrow \varinjlim \text{Hom}(Y, X_i). \quad (28)$$

By Yoneda's Lemma and the fact that  $\text{Hom}(- \otimes X)$  commutes with colimits, we have  $\text{Hom}(\mathbf{X}, \mathbf{Y}) \cong \text{Hom}(\mathbf{L}_{\mathbf{X}}, \mathbf{L}_{\mathbf{Y}})$ . In fact, from definition,  $\mathbf{L}_{\mathbf{X}}$  is isomorphic to  $\varinjlim_i \mathbf{L}_{X_i}$  in  $\text{Fun}(\mathbf{C}^{\text{op}}, \text{Set})$ . Therefore

$$\begin{aligned} \text{Hom}(\mathbf{L}_{\mathbf{X}}, \mathbf{L}_{\mathbf{Y}}) &= \varprojlim_i \text{Hom}(\mathbf{L}_{X_i}, \mathbf{L}_{\mathbf{Y}}) \\ (\text{by Yoneda's Lemma}) &= \varprojlim_i \mathbf{L}_{\mathbf{Y}}(X_i) \\ &= \varprojlim_i (\varinjlim_j \text{Hom}(X_i, Y_j)). \end{aligned}$$

The category  $\text{Ind-C}$  is closed under filtering direct limits (cf. [2, Appendix. 4.4] or [1, I.8]).

Note however that the direct limit computed in  $\mathbf{C}$  (if it exists) is generally different from the one computed in  $\text{Ind-C}$ .

**3.2. Extension of functors.** Let  $F : \mathbf{C} \longrightarrow \mathbf{D}$  be a functor. A functor  $\text{ind-F} : \text{Ind-C} \longrightarrow \text{ind-D}$  is defined as follows:

$$\text{ind-F}(\{X_i\}_{i \in \mathbf{I}}) := \{F(X_i)\}_{i \in \mathbf{I}}. \quad (29)$$

The action of  $\text{ind-F}$  on hom-sets is defined in a straightforward manner.

**3.3. Ind-category for abelian categories.** Assume now that  $\mathbf{C}$  is abelian. Then the functor  $\mathbf{L}_{\mathbf{X}}$  is exact for any ind-object  $\mathbf{X} = \{X_i\}_{i \in \mathbf{I}}$ . Indeed, in the category of sets, the filtering direct limits preserves left exact sequences, hence for a left exact sequence in  $\mathbf{C}^{\text{op}} : 0 \longrightarrow Y \longrightarrow Y' \longrightarrow Y''$ , we have the following left exact sequences

$$0 \longrightarrow \text{Hom}(Y, X_i) \longrightarrow \text{Hom}(Y', X_i) \longrightarrow \text{Hom}(Y'', X_i)$$

$i \in \mathbf{I}$ . Taking limit we have

$$0 \longrightarrow \mathbf{L}_{\mathbf{X}}(Y) \longrightarrow \mathbf{L}_{\mathbf{X}}(Y') \longrightarrow \mathbf{L}_{\mathbf{X}}(Y'').$$

Conversely, let  $\mathbf{L}$  be a functor  $\mathbf{C}^{\text{op}} \longrightarrow \mathbf{Ab}$ . By Yoneda's Lemma, for any  $X \in \mathbf{C}$ ,  $\text{Hom}(\text{Hom}(- \otimes X), \mathbf{L}(-)) \cong \mathbf{L}(X)$ . Consider the system  $C_{\mathbf{L}} := \{(X, \eta) | X \in \mathbf{C}, \eta \in \mathbf{L}(X)\}$ . Morphism



$(X, \eta) \longrightarrow (X', \eta')$  are those from  $\text{Hom}(X, X')$ , which commute with  $\eta, \eta'$ , in the sense that  $L(f)(\eta') = \eta$ . Then  $L$  is isomorphic to the functor

$$Y \longmapsto \varinjlim_{(X, \eta) \in C_L} \text{Hom}(Y, X).$$

If  $L$  is left exact then  $C_L$  is a filtering system. In fact, given  $(X, \eta)$  and  $(X', \eta')$ , since left exact functors preserve direct sums, we can take the object  $(X'', \eta'')$  to be  $(X \oplus X', \eta \oplus \eta')$ . For any two morphisms  $f, f' : (X, \eta) \longrightarrow (X', \eta')$ , let  $g : X' \longrightarrow Y$  be the coequalizer of  $f$  and  $f'$ . Since  $F$  is left exact (from  $C^{\text{op}}$  to  $\text{Ab}$ ), we can take  $\zeta := L(g)^{-1}(\eta')$ . Then  $(Y, \zeta)$  is the required pair with  $g$  equalizing  $f$  and  $f'$ .

Thus, a functor from  $C^{\text{op}} \longrightarrow \text{Ab}$  is left exact if and only if it has the form as in (28). The category of left exact functor  $\text{Lex}(C^{\text{op}}, \text{Ab})$  is naturally equivalent with  $\text{Ind-C}$  (cf [2, Appendix 4.5]). We know that the category  $\text{Lex}(C^{\text{op}}, \text{Ab})$  is a Grothendieck category, i.e. complete, cocomplete with a generator and filtering limits preserve exact sequences. In such a category injective envelopes exist and an injective cogenerator exists (cf. [10, Chapter II] or [13, Chapter V, X]).

**3.4. Extension of monoidal structures.** Assume now that  $C$  is a monoidal category. Thus, we have a bifunctor  $\odot : C \times C \longrightarrow C$ , which induces a bifunctor  $\odot : \text{ind-C} \times \text{ind-C} \longrightarrow \text{ind-C}$ . Explicitly, for ind-objects  $X = \{X_i\}$  and  $Y = \{Y_j\}$ , we set

$$X \odot Y := \varinjlim_i \varinjlim_j X_i \odot Y_j. \quad (30)$$

It is easy to see that this functor define a monoidal structure on  $\text{Ind-C}$ , with the unit object being the unit object in  $C$ . In fact, we have, for any ind-object  $X$ :

$$X \odot I = \varinjlim_i X_i \odot I = \varinjlim_i X_i = X.$$

Assume now that  $C$  is abelian. Since directed limits preserve exact sequences, the tensor product in  $\text{Ind-C}$  is left (right) exact whenever the tensor product in  $C$  is. Indeed, by Lemma 2.2, any left exact sequence  $0 \longrightarrow X \longrightarrow X' \longrightarrow X''$  can be “uniformly” represented by a filtering system

$$0 \longrightarrow X_i \longrightarrow X'_i \longrightarrow X''_i, \quad i \in I.$$

Thus, assuming that the tensor product on  $C$  is left exact, for any object  $Y_j$  of  $C$ , the sequence

$$0 \longrightarrow X_i \odot Y_j \longrightarrow X'_i \odot Y_j \longrightarrow X''_i \odot Y_j, \quad i \in I, j \in J,$$

is exact. Since filtering limits preserve exact sequences, we have a left exact sequence

$$0 \longrightarrow X \odot Y_j \longrightarrow X' \odot Y_j \longrightarrow X'' \odot Y_j, j \in J.$$

Taking the limit after  $j$ , we obtain a left exact sequence

$$0 \longrightarrow X \odot Y \longrightarrow X' \odot Y \longrightarrow X'' \odot Y.$$

In particular, the tensor product on  $\text{Ind-C}$  is exact if the tensor product on  $C$  is.

**3.5. An embedding theorem for small abelian monoidal categories with exact tensor product.** Let  $C$  be a small, abelian monoidal category with an exact tensor product. Set  $A := \text{Ind-C}$ . Then we see in the previous section that  $A$  is a monoidal Grothendieck category with an exact tensor product. Let  $J$  be an injective cogenerator in  $A$ , which exists due to the fact that  $A$  is a Grothendieck category.

Let  $R := \text{End}(J)$ . Consider the functor

$$\text{Hom}(-, J) : A^{\text{op}} \longrightarrow {}_R\text{Mod}.$$

Since  $J$  is injective, the functor is exact and since  $J$  is a cogenerator, the functor is faithful. Moreover, the functor is full whenever  $X$  is a submodule of  $J^{\oplus n}$ ,  $n < \infty$  (cf. [10, IV.4.1]). The object  $J$  can be chosen so that every object of  $\mathbf{C}$  satisfies this condition. Therefore, the embedding  $\mathbf{C}^{\text{op}} \rightarrow {}_R\mathbf{Mod}$  is exact and full.

The tensor product on  $\mathbf{A}$  induces a bifunctor on a subcategory of  ${}_R\mathbf{Mod}$ , which contains  $R$  – a progenerator of  ${}_R\mathbf{Mod}$ . Notice that  $\mathbf{Hom}_{\mathbf{A}^{\text{op}}}(J, J) = {}_R\mathbf{Hom}(R, R)$ . Therefore we can extend the tensor product, which is considered as a functor on the full subcategory of  ${}_R\mathbf{Mod}_R$ , consisting of one object  $R$ , to a colimit preserving functor on the whole category  ${}_R\mathbf{Mod}$  (cf. [10, V.5.2, p106]). The explicit construction is given as follows.

First, we define the tensor product on the direct sum of  $R$ . For any sets  $S, T$ ,  $R^S \square R^T = (R \odot R)^{S \times T}$ . Then, for any module  $M$ , take a resolution

$$R^S \xrightarrow{f} R^T \xrightarrow{g} M \rightarrow 0$$

and define  $R^U \square M$  to be the cokern of  $R^U \square f$ :

$$R^U \square R^S \rightarrow R^U \square R^T \rightarrow R^U \square M \rightarrow 0$$

Analogously, we define  $M \square R^U$  and then  $M \square N$ . Lemma V.5.2.1 of [10] ensures that the above construction does not depend on the choice of resolution. The associator is defined first on the direct sums of  $R$  and then projected on the other objects.

From the construction of  $\square$ , we see that if  $M$  and  $N$  are finitely presented modules then

$$M \odot N \cong M \square N.$$

On the other hand, we know that if an object  $X$  of  $\mathbf{C}$  has a resolution of the form  $0 \rightarrow X \rightarrow J^S \rightarrow J^T$  where  $S, T$  are finite sets, then the  $R$ -module  $M = \mathbf{Hom}(X, J)$  is a finitely presented  $R$ -module.

This condition may not be satisfied for any injective cogenerator. However, it can be achieved by increasing the cogenerator. We take the direct sum of all objects from  $\mathbf{C}$  and then take its injective envelope. Denote the object obtained by  $J_1$ . Then  $J \oplus J_1$  is also an injective cogenerator, in which every object of  $\mathbf{C}$  can be embedded. For any  $X \in \mathbf{C}$ , let  $i_X$  be an embedding in  $J \oplus J_1$  and let  $X'$  be the cokern of  $i_X$ , i.e., we have an exact sequence  $0 \rightarrow X \rightarrow J \oplus J_1 \rightarrow X' \rightarrow 0$ . Now, let  $J_2$  be the injective envelope of the direct sum of all  $X'$  where  $X$  runs in  $\mathbf{C}$ . Let  $\bar{J} := J \oplus J_1 \oplus J_2$ . Then for any  $X \in \mathbf{C}$ , we have a resolution by  $\bar{J}$ :

$$0 \rightarrow X \hookrightarrow \bar{J} \twoheadrightarrow J \oplus X' \oplus J_2 \hookrightarrow \bar{J} \oplus \bar{J}.$$

Since  $\square$  preserves colimit and since any module is a filtering direct limit of finite presented modules (cf. [13, I.5]), we have, for any  $R$ -module  $M = \varinjlim_i M_i$ ,  $M_i$  are finitely presented  $R$ -modules,

$$I \square M \cong I \square \varinjlim_i M_i \cong \varinjlim_i (I \square M_i) \cong \varinjlim_i M_i = M.$$

Thus,  $I$  is the unit object in  ${}_R\mathbf{Mod}$  with respect to the tensor product  $\square$ .

Applying the result of the previous section, we have a monoidal functor

$$\omega : {}_R\mathbf{Mod} \rightarrow {}_R\mathbf{Mod}_R, \quad M \mapsto M \square R,$$

which is a right exact embedding. Compose  $\omega$  with the functor  $\mathbf{Hom}(-, \bar{J})$ , we get a right exact functor from  $\mathbf{A}^{\text{op}}$  to  ${}_R\mathbf{Mod}_R$ , whose restriction on  $\mathbf{C}^{\text{op}}$  is a right exact monoidal embedding. The last functor is given by

$$X \mapsto \mathbf{Hom}(X \odot J, J).$$

If we start instead with  $\mathbf{C}^{\text{op}}$  then the above discussion give us a monoidal right exact embedding from  $\mathbf{C}$  to  ${}_R\mathbf{Mod}_R$ . Thus we have proved

**Theorem 3.2.** *Let  $\mathbf{C}$  be a small abelian monoidal category with the tensor product being exact. Then  $\mathbf{C}$  admits a right exact monoidal embedding into the category  ${}_R\mathbf{Mod}_R$  for some ring  $R$ . The functor is given explicitly by  $X \longrightarrow \text{Hom}(J, X \odot J)$  for a suitably chosen injective cogenerator  $J$  in  $\mathbf{Ind-C}$ .*

In particular, if  $\mathbf{C}$  is an abelian rigid monoidal category then the tensor product is exact and the theorem above applies. In this case, the embedding is exact. Indeed, let  $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$  be an exact sequence. Applying the left dual functor, we have an exact sequence  $0 \longrightarrow Z^* \longrightarrow Y^* \longrightarrow X^* \longrightarrow 0$ . Let  $F$  denote the embedding functor. Then we have a right exact sequence of  $R$ -modules:

$$F(Z^*) \longrightarrow F(Y^*) \longrightarrow F(X^*) \longrightarrow 0 \quad (31)$$

Since  $F$  is a monoidal functor,  $F(X^*) \cong F(X)^* = \text{Lhom}(F(X), R)$ . Therefore  $F(X) \cong \text{Rhom}(F(X^*), R)$ . Applying the left exact functor  $\text{Hom}(-, R)$ , we obtain a left exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z).$$

Since the category of bimodules over a ring is cocomplete, the embedding can be extended to a functor  $\text{ind-F} : \text{ind-A} \longrightarrow {}_R\mathbf{Mod}_R$ , which is also exact and monoidal. Explicitly, this functor has the form, for  $\mathbf{X} = \{X_i\}_{i \in I}$ :

$$\text{ind-F}(\mathbf{X}) = \varinjlim_i \text{Hom}(J, X_i \odot J). \quad (32)$$

**Theorem 3.3.** *Let  $\mathbf{C}$  be a small abelian monoidal rigid category. Then  $\mathbf{C}$  admits an exact monoidal embedding in to the category of bimodules over a ring. Further, the embedding is extendable to an exact embedding of the category  $\text{ind-C}$ , which commutes with colimits.*

*Proof.* What remains to be proved is that the functor  $\text{ind-F}$  is faithful or equivalently that  $\text{ind-F}(\mathbf{X}) \neq 0$  whenever  $\mathbf{X} \neq 0$ .

First, we remark that, since  $\text{ind-F}$  is exact,  $\text{ind-F}(\mathbf{X}) \not\cong 0$  whenever  $\mathbf{X}$  possesses a subobject (or a quotient object)  $\mathbf{Y}$ , with  $\text{ind-F}(\mathbf{Y}) \not\cong 0$ .

The following fact in  $\mathbf{Ind-C}$  is well-known (cf. [7, Cor. II.3.2]). If and ind-object  $\mathbf{X}$  is a subobject of an object  $X \in \mathbf{C}$ , then  $\mathbf{X}$  contains a subobject  $Y \in \mathbf{C}$ . Indeed, let  $i : \mathbf{X} \longrightarrow X$  be a monomorphism in  $\mathbf{Ind-C}$  and  $j : Y \longrightarrow \mathbf{X}$  be a non-zero morphism,  $Y \in \mathbf{C}$ , then  $i \circ j : Y \longrightarrow X$  is non-zero and is a morphism in  $\mathbf{C}$ , for  $\mathbf{C}$  is a full subcategory of  $\mathbf{Ind-C}$ .

A direct consequence of this fact and the preceding remark is that the image under  $\text{ind-F}$  of any non-zero ind-object, which is a subobject of an object from  $\mathbf{C}$ , is non-zero.

Let now  $\mathbf{X} \in \mathbf{Ind-C}$  be a non-zero object. There exists a non-zero morphism  $j : Y \longrightarrow \text{Lhom}(\mathbf{X}, I)$ ,  $Y \in \mathbf{C}$ . Since

$$\text{Hom}(\mathbf{X}, {}^*Y) \cong \text{Hom}(Y \odot \mathbf{X}, I) \cong \text{Hom}(Y, \text{Lhom}(\mathbf{X}, I)),$$

there exists a non-zero morphism  $k : \mathbf{X} \longrightarrow {}^*Y$ , corresponding to  $j$  in the above isomorphisms.  $\text{Im}k$  is a non-zero subobject of  ${}^*Y$ , hence  $\text{ind-F}(\text{Im}k) \not\cong 0$ , consequently,  $\text{ind-F}(\mathbf{X}) \not\cong 0$ . ■

## 4. SEMISIMPLE ABELIAN RIGID MONOIDAL CATEGORIES

In this section we consider a simple case, where the construction in the previous section can be explicitly given. Let  $\mathbf{C}$  be a semisimple rigid monoidal category. Thus, as abelian category,  $\mathbf{C}$  is characterized by its simple objects  $X_i, i \in \mathcal{I}$ , and the ring  $R_i = \text{End}(X_i)$ , where  $\mathcal{I}$  is a set. The category  $\mathbf{A} = \text{Ind-}\mathbf{C}$  is easy to characterize. Each object of  $\mathbf{A}$  is a direct sum of copies of  $X_i, i \in \mathcal{I}$ . An injective cogenerator can be chosen to be  $J = \bigoplus_{i \in \mathcal{I}} X_i$ . Our embedding is then

$$X \longmapsto \text{Hom}\left(\bigoplus_j X_j, X_i \odot \left(\bigoplus_k X_k\right)\right) \cong \prod_j \bigoplus_k \text{Hom}(X_j, X_i \odot X_k), \quad (33)$$

in the category of  $R - R$ -bimodules, where  $R = \text{End}(J) \cong \prod_i R_i$ . Each  $R_i$  being an endomorphism ring of a simple object is a skew-field (non-commutative field).

For any object  $X \in \mathbf{C}$ , we have  $\text{End}(X) \cong \text{Hom}(I, X \otimes X^*)$ . If  $I$  is a simple object, then the dimension of  $\text{End}(X)$  over  $K = \text{End}(I)$  is equal to the number of copies of  $I$  in the decomposition of  $X \otimes X^*$ . Through the above embedding,  $I$  is mapped to the ring  $R$  and  $X \otimes X^*$  is mapped to a bimodule over  $R$  which is projective of finite rank when considered as left or right  $R$ -module. Therefore  $X \otimes X^*$  can contain only finitely many copies of  $I$  in its decomposition into simple objects. Consequently,  $\text{End}(X)$  is finite dimension over  $K$ .

**Proposition 4.1.** *Let  $\mathbf{C}$  be a semisimple rigid monoidal category with a simple unit object. Then for any object  $X \in \mathbf{C}$ ,  $\text{End}(X)$  is finite dimensional over  $\text{End}(I)$ , in particular, objects of  $X$  are direct sums of simple objects.*

From now on we shall assume that  $\mathbf{C}$  is a semisimple rigid monoidal category with a simple unit object. Let  $V_i^j := \text{Hom}(X_j, X_i \odot X_k)$  then  $V_i^j$  is an  $R - R$ -bimodule, whose left action is induced by the left action of  $R_k$  and whose right action is induced by the left action of  $R_j$ . Since  $R_j$  and  $R_k$  are skew-fields,  $V_i^j$  becomes a left and right vector space over  $R_k$  and  $R_j$ , respectively.

Let  $c_{ik}^j$  be the multiplicity of  $X_j$  in  $X_i \odot X_k$ . Then  $\dim_{R_j} V_i^j = c_{ik}^j$ . On the other hand, since  $\mathbf{C}$  is rigid, we have

$$\text{Hom}(X_j, X_i \odot X_k) \cong \text{Hom}(X_i^* \odot X_j, X_k). \quad (34)$$

Therefore  $\dim_{R_k} V_i^j = c_{i^*j}^k$ . In particular,  $V_i^j$  is finite dimensional over  $R_j$  and  $R_k$ . Moreover, if we fix  $i, j$  and let  $k$  run in  $\mathcal{I}$  then (34) shows that there are only finitely many  $k$  for which  $V_i^j$  is non-zero. Analogously, for  $i, k$  fixed, there are only finitely many  $j$  for which  $V_i^j$  is non-zero. We also notice that  $V_i^j \otimes V_l^m = 0$  unless  $k = n$ .

Since  $X_i$  is rigid, its image in  ${}_R\text{Mod}_R$  is projective of finite rank when considered as a left of a right  $R$ -module, thus, particularly finitely presented. Since the tensor product with a finitely presented module commute with direct product, (cf. [13, Lemma I.13.2]), we have

$$\begin{aligned} \left(\prod_j \bigoplus_k V_i^j\right) \otimes \left(\prod_m \bigoplus_n V_l^m\right) &\cong \prod_j \bigoplus_k \left(V_i^j \otimes \left(\prod_m \bigoplus_n V_l^m\right)\right) \\ &= \prod_j \bigoplus_k \left(V_i^j \otimes \left(\bigoplus_n V_l^m\right)\right) \\ &= \prod_j \bigoplus_n \left(\bigoplus_k V_i^j \otimes V_l^k\right). \end{aligned}$$

Notice that in the last term of the above equations, the index  $k$  runs in a finite set for each fixed  $j$  and  $n$ . Thus, we can interpret the bimodule image of any object of  $\mathbf{C}$  as an infinite

matrix in which there is only finite number of non-zero element on each row or each column. The tensor product is given in terms of matrix multiplication. Dual objects correspond to transposed matrices.

The discussion above allows us to give an estimation on the dimension of the space  $\text{End}(X^{\odot n})$  over  $K = \text{End}(I)$  assuming that  $I$  is simple and  $R_i = K$  for all  $i \in \mathcal{I}$ . Notice that although the two actions of  $K$  on  $\text{Hom}(X, Y)$  may be different, the dimension of  $\text{Hom}(X, Y)$  over  $K$  with respect to these actions are equal (whenever they are finite). Thus, it is meaningful to speak of the dimension over  $K$ . In our case,  $\text{End}(X^{\odot n}) \cong \text{Hom}(I, X^{\odot n} \odot X^{\odot n*})$ . Since  $I$  is simple, the dimension of  $\text{End}(X^{\odot n})$  is equal to the number of copies of  $I$  in the decomposition of  $\text{Hom}(I, X^{\odot n} \odot X^{\odot n*})$ . We want to show that this dimension does not exceed  $d^n$  for some positive  $d$  depending only on  $V$ . Embedding  $\mathcal{C}$  into  ${}_R\text{Mod}_R$  as above, we see that this dimension can not exceed the number of copies of  $R$  in the image of  $X^{\odot n} \odot X^{\odot n*}$ . Let  $V$  be the matrix representing the image of  $X$ , then  $V^n \cdot {}^t(V^n)$  represents the image of  $X^{\odot n} \odot X^{\odot n*}$ .  $R$  itself is represented by the identity matrix. Therefore, the number of copies of  $R$  in the bimodule  $V^n \cdot {}^t(V^n)$  is equal to the minimal among the dimension over  $K$  of the sub-bimodules lying in the diagonal of  $V^n \cdot {}^t(V^n)$ .

Since  $X$  is rigid,  $V$  represents a projective module of finite rank over  $R$ , therefore, the sum of dimension of  $V_k^j$  on each row or each column should be uniquely bounded by a certain number  $d$ . Then, the same holds for the matrix  $V^n \cdot {}^t(V^n)$  with  $d$  replaced by  $d^{2n}$ . In particular, the dimension of  $(V^n \cdot {}^t(V^n))_i^i$  should not exceed  $d^{2n}$ . Thus, we have proved

**Theorem 4.2.** *Let  $\mathcal{C}$  be a semisimple abelian rigid monoidal category with simple unit object, whose endomorphism ring is denoted by  $K$ . Assume that for any simple object, its endomorphism ring is isomorphic to  $K$ . Then for any object  $X$ , there exists a positive number  $d$ , such that the dimension over  $K$  of  $\text{End}(X^n)$  does not exceed  $d^n$ .*

*Remark.* The condition  $R_i = K, \forall i \in \mathcal{I}$  can be replaced by the condition that the dimension of  $R_i$  over  $K$  is globally bounded by a number  $c$ . In this case,  $d$  should be replaced by  $dc^2$ .

Theorem 4.2 has the following important consequence.

**Corollary 4.3.** *Assume that  $\mathcal{C}$  satisfies the condition of Theorem 4.2 and that, moreover  $\mathcal{C}$  is symmetric. Then, if  $\text{char } K = 0$ , we can modify the symmetry on  $\mathcal{C}$  so that for any object  $X$  of  $\mathcal{C}$ , there exists an integer  $n$ , for which  $\Lambda_n(X)$  – the  $n$ -th antisymmetric tensor power of  $X$  is zero. Consequently  $\mathcal{C}$  is Tannakian.*

We give here only a sketch of the proof. A detailed proof will be given elsewhere.

Given a symmetry of  $\mathcal{C}$ , we define for each object  $X$  its categorical dimension to be the morphism  $I \xrightarrow{\text{db}} X \odot X^* \xrightarrow{\tau} X^* \odot X \xrightarrow{\text{ev}} I$ , an element of  $K = \text{End}(I)$ . This dimension is an additive and tensor-multiplicative function on  $X$ . Since the category is semisimple, the dimension of a simple object is non-zero.

On the other hand, the symmetry induces a representation of  $k[\mathfrak{S}_n]$  in  $\text{End}(X^{\odot n})$  for any object  $X$ ,  $\mathfrak{S}_n$  is the symmetric group. Theorem 4.2 ensures that starting from some  $n$ , the representation is not faithful. That means some subobject of  $X^{\odot n}$  should be zero; its dimension is therefore also zero. This implies that the categorical dimension of  $X$  should be an integer. Thanks the semisimplicity we can modify the symmetry on  $\mathcal{C}$  so that the dimension of a simple object is a positive integer, hence so is the dimension of any object. Then we are done by [3, Theorem 7.1]. ■

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